LECTURE XXIII

1. Recap and Special Series

1.1. Definition of Sequence of Partial Sums, S_N. Given a series $\sum_{n=1}^{\infty} a_n$, its N^{th} partial sum is defined as

$$S_N = \sum_{n=1}^{N} a_n$$

One should view S_N as a sequence in N.

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &. \\ &. \\ S_N &= a_1 + \cdots + a_N \end{aligned}$$

1.2. Special Series.

Example 1. Geometric Series with ratio r:

$$a + ar + ar^2 + ar^3 + \cdots = a \sum_{n=1}^{\infty} r^{n-1}$$

Partial sum

$$S_N = a \sum_{n=1}^{N} r^{n-1} = a + ar + ar^2 + \cdots + ar^{N-1}$$

Multiply by r to get

$$rS_N = ar + ar^2 + \cdots + ar^{N-1} + ar^N$$

Then via cancellation of common terms, we must have

$$S_N - rS_N = a - ar^N = a\left(1 - r^N\right)$$

Solving for

$$S_N = \frac{a\left(1 - r^N\right)}{1 - r}$$

This formula is true for any r. For example, take $r = 2$ with $a = 2$ for 5 terms, we have essentially S_5, i.e.

$$S_5 = 2 + 4 + 8 + 16 + 32 = \sum_{n=1}^{5} 2\,(2)^{n-1} = \frac{2\left(1 - 2^5\right)}{1 - 2} = 64$$

Is it true?

$$S_5 = 64$$

by brute force addition.

This formula becomes amazing when $|r| < 1$ because,

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty} \sum_{n=1}^{N} ar^{n-1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a\left(1 - r^N\right)}{1 - r} = \frac{a}{1 - r}$$

providing a formula for the sum of an infinite geometric series with ratio $|r| < 1$.

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Example 2. Telescoping series.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Its N^{th} partial sum is

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1}$$

with all middle terms cancelling out. Knowing that the limit of S_N is the sum of the series, it's easy to see that

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

2. Divergence Test

When the terms in a series (the sequence being added) becomes too big as n increases, the series may fail to converge.

$$\sum_{n=1}^{\infty} \frac{n+1}{n}$$

One way to show that this series diverges is to compare its partial sum to known ones. One trivial partial sum you can use is simply the sum of all 1's.

$$S_N = \sum_{n=1}^{N} \frac{n+1}{n} \geq \sum_{n=1}^{N} 1 = N$$

Therefore,

$$\lim_{N\to\infty} S_N \geq \lim_{N\to\infty} N = \infty$$

which shows divergence.

Another way to deduce divergence is to check the limit of the sequence being added.

Theorem 3. *(The n^{th} term test for divergence). If $\lim_{n\to\infty} a_n$ fails to exist or is different from zero, then $\sum_{n=1}^{\infty} a_n$ diverges (not necessarily to ∞).*

The logically equivalent statement of this theorem is a powerful statement as well.

Theorem 4. *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.*

Remark. Poll: True or false. If $a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Example. False! Counterexample: $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges, yet indeed $\frac{1}{n} \to 0$. What should you conclude about $\sum_{n=1}^{\infty} a_n$ then, if $a_n \to 0$? ABSOLUTELY NOTHING!

How do we use the two above powerful theorems?

Example 5.

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

diverges because the sequence inside

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1}$$

does not exist (it alternates). If the limit of the sequence (being added) fails to exists, it satisfies the condition of Theorem 3 and therefore, the conclusion of Theorem 3 follows. The series diverges.

Example 6.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

diverges because by theorem 3

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

a limit different from zero.

3. Operations of Multiple Series

If $\sum a_n = A$ and $\sum b_n = B$, then

1. Sum/Difference rule: $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$.
2. Constant Multiple: $\sum k a_n = k \sum a_n = kA$.

Examples are easy to follow. A complicated looking series usually requires few steps of simplification before using the above rules. For example,

Example.

$$\sum_{n=1}^{\infty} \frac{2^{n-1} - 6}{6^n}$$

The very first thing to do is to divide.

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{2^{n-1}}{6^n} - \frac{6}{6^n} \right) &= \sum_{n=1}^{\infty} \left(\frac{2^n \cdot 2^{-1}}{6^n} - \frac{1}{6^{n-1}} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2} \left(\frac{1}{3} \right)^n - \frac{1}{6^{n-1}} \right) \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{6} \right)^{n-1} \\
&= \frac{1}{2} \left(\frac{1/3}{1 - (1/3)} \right) - \frac{1}{1 - (1/6)} \\
&= -\frac{19}{20}
\end{aligned}$$

Poll: Can two divergent series $\sum a_n$ and $\sum b_n$ yield $\sum (a_n + b_n)$ being convergent?

Solution. Yes. Simply choose $b_n = -a_n$.

$$\sum (a_n + b_n) = \sum (a_n - a_n) = 0$$

4. One Important Characteristic of a Series

Poll: Will the first n terms of a series affect the convergence behaviour of the series?

Solution. No. They add up to a finite number. There are still infinite number of terms yet to come. In essence,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_N + \sum_{n=N+1}^{\infty} a_n$$

Rearranging, putting the tail of a series on one side, we see

$$\sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - (a_1 + a_2 + \cdots + a_N)$$

Now, if the original series converges, then $\sum_{n=1}^{\infty} a_n = L$ for some number L. Then certainly, the tail converges to $L - (a_1 + a_2 + \cdots + a_N)$. On the other hand, if the tail converges,

$$\sum_{n=N+1}^{\infty} a_n = K$$

then

$$\sum_{n=1}^{\infty} a_n = K + (a_1 + a_2 + \cdots + a_N)$$

which is finite as well, showing convergences.

Therefore, only the tails of a series affects its convergence.